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ON AXIALLY UNIFORM STRESS AND STRAIN IN AXIALLY HOMOGENEOUS CYLINDRICAL SHELLS

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INTRODUCTION

IN WHAT follows we first state the differential equations of linear theory for axially uniform stress and strain in axially homogeneous cylindrical shells, as a special case of the system of general shell equations which have recently been formulated by Schaefer [6], Günther [1] and the present author [3]. The dominant characteristics of these equations are the inclusion of moments turning about the normals to the middle surface of the shell, together with the effect of transverse shear deformation, and the ensuing clarity and simplicity of the static geometric duality, involving equilibrium, compatibility and constitutive equations. We then show that the complete system uncouples, for rather general classes of stress strain relations, into two separate systems of equations. One of these, which is considered further in what follows, deals with the problem of St. Venant torsion of thin walled tubes as a problem of the two-dimensional theory of elastic cylindrical shells. We have previously considered this problem within the framework of the two-dimensional theory of thin shells [2] and have shown in particular (1) the importance of retaining certain *small* terms in one of the equations of moment equilibrium, and (2) the possibility of obtaining an explicit torque-twist relation and explicit expressions for stress resultants and couples which include, as limiting special cases, the wellknown relations of Bredt for closed tubes and the completely different relations for open tubes as stated by Prescott.

Our present consideration of the problem of torsion, in addition to being somewhat more general than our earlier approach, is simpler and more transparent than the earlier approach, reflecting the extent to which all linear shell theory has become simpler and more transparent in the course of the last ten years.

DIFFERENTIAL EQUATIONS FOR AXIALLY UNIFORM STRESS AND STRAIN IN CYLINDRICAL SHELLS

We take as curvilinear coordinates on the middle surface of the shell circumferential arc length s and axial distance z. We then have as coefficients of the linear element of the shell $\alpha_s = \alpha_z = 1$ and as curvature measures $1/R_{ss} \equiv 1/R = -\phi'$ and $1/R_{sz} = 1/R_{zz} = 0$ where $\phi = \phi(s)$ is the tangent angle to the meridians of the shell, in accordance with Fig. 1, and where primes indicate differentiation with respect to s.

We take the differential equations of this theory for general orthogonal coordinates as recently summarized [4, 5] (setting $\xi_1 = s$ and $\xi_2 = z$) and furthermore assume that

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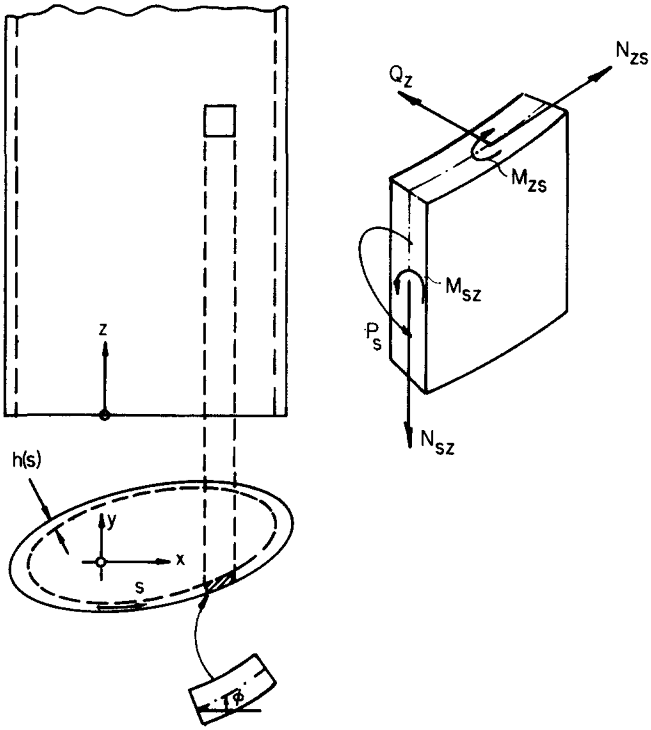

FIG. 1. Axially uniform cylindrical shell and element of shell showing stress resultants and couples for the problem of torsion.

stress resultants and stress couples, as well as the corresponding strain measures are independent of the axial coordinate z. Omitting for simplicity's sake surface force and moment load intensity terms, we then have six equilibrium equations and six compatibility equations of the following form

$$N'_{ss}+\frac{Q_s}{R}=0, \qquad \kappa'_{zz}-\frac{\lambda_z}{R}=0 \tag{1a, b}$$

$$N'_{sz}=0, \qquad \kappa'_{zs}=0 \tag{2a, b}$$

$$Q'_s-\frac{N_{ss}}{R}=0, \qquad \lambda_z+\frac{\kappa_{zz}}{R}=0 \tag{3a, b}$$

$$M'_{ss}-Q_s=0 \qquad \varepsilon'_{zz}-\lambda_z=0 \tag{4a, b}$$

$$M'_{sz}-Q_z-\frac{P_s}{R}=0, \qquad \varepsilon'_{zs}-\lambda_s+\frac{\gamma_z}{R}=0 \tag{5a, b}$$

$$P'_s+N_{sz}-N_{zs}+\frac{M_{sz}}{R}=0, \qquad \gamma'_z+\kappa_{zs}-\kappa_{sz}-\frac{\varepsilon_{zs}}{R}=0. \tag{6a, b}$$

Equations (1) to (6) are complemented by constitutive equations which are here taken to be of the form

$$N_{ss} = \frac{\partial U}{\partial \varepsilon_{ss}}, \quad N_{sz} = \frac{\partial U}{\partial \varepsilon_{sz}}, \quad \ldots, \quad Q_z = \frac{\partial U}{\partial \gamma_z}, \quad \ldots, \quad M_{ss} = \frac{\partial U}{\partial \kappa_{ss}}, \quad \ldots, \quad P_z = \frac{\partial U}{\partial \lambda_z} \tag{7}$$

with U a given function of the indicated twelve arguments.

Inspection indicates that the system (1) to (6) consists of two separate sets of equations. These are equations (1), (3) and (4), and equations (2), (5) and (6), respectively. While, in general, the system (1), (3) and (4) will be coupled to the system (2), (5) and (6) because of the form of the constitutive equations (7) there will be no such coupling for quite large classes of constitutive equations, i.e., for the class of constitutive equations for which

$$U = U_1(\epsilon_{ss}, \epsilon_{zz}, \gamma_s, \kappa_{ss}, \kappa_{zz}, \lambda_z) + U_2(\epsilon_{sz}, \epsilon_{zs}, \gamma_z, \kappa_{sz}, \kappa_{zs}, \lambda_s).$$

Equations (1) to (7) are to be considered in conjunction with appropriate boundary or transition conditions for sections s = const. of the shell. These may pertain to forces and moments and/or to the associated components of translational and rotational displacement. For an interpretation of the latter in terms of components of strain, use is made of strain displacement relations of the form

$$\begin{gathered} \varepsilon_{ss} = u_{s,s} + \frac{w}{R}, \quad \varepsilon_{sz} = u_{z,s} - \omega, \quad \varepsilon_{zs} = u_{s,z} + \omega, \quad \varepsilon_{zz} = u_{z,z} \\ \gamma_s = \phi_s + w_{,s} - \frac{u_s}{R}, \quad \gamma_z = \phi_z + w_{,z}, \quad \lambda_s = \omega_{,s} + \frac{\phi_z}{R}, \quad \lambda_z = \omega_{\,z} \\ \kappa_{ss} = \phi_{s,s}, \quad \kappa_{sz} = \phi_{z,s} - \frac{\omega}{R}, \quad \kappa_{zs} = \phi_{s,z}, \quad \kappa_{zz} = \phi_{z,z}. \end{gathered} \tag{8}$$

In these the nature of the z-dependence of the components of displacement is limited by the condition that all components of strain must be independent of z.

DIFFERENTIAL EQUATIONS FOR TORSION

We have as expression for the torque T acting over the cross sections of the shell the integral

$$T = \int [M_{zs} - (N_{zs} \sin \phi + Q_z \cos \phi)\, x + [N_{zs} \cos \phi - Q_z \sin \phi)\, y]\, \mathrm{d}s. \tag{9}$$

In order to evaluate (9) we make use of the equilibrium equations

$$N'_{sz} = 0, \qquad M'_{sz} - Q_z + \phi' P_s = 0, \qquad P'_s + N_{sz} - N_{zs} - \phi' M_{sz} = 0 \tag{10}$$

and of the compatibility equations

$$\kappa'_{zs} = 0, \qquad \varepsilon'_{zs} - \lambda_s - \phi' \gamma_z = 0, \qquad \gamma'_z + \kappa_{zs} - \kappa_{sz} + \phi' \varepsilon_{zs} = 0 \tag{11}$$

together with constitutive equations which for definiteness sake are now taken in the form

$$
\begin{aligned}
\varepsilon_{zs} &= AN_{zs}+A_*N_{sz}, \qquad \varepsilon_{sz} = AN_{sz}+A_*N_{zs}, \qquad \gamma_z = A_Q Q_z \\
M_{zs} &= D\kappa_{zs}+D_*\kappa_{sz}, \qquad M_{sz} = D\kappa_{sz}+D_*\kappa_{zs}, \qquad P_s = D_P\lambda_s
\end{aligned} \tag{12}
$$

where A, A_*, A_Q, D, D_* and D_P are given functions of s.

We furthermore assume that the vanishing of all the stress resultants and couples in (1a), (3a) and (4a) implies, in conjunction with the associated constitutive equations, the strain conditions

$$\varepsilon_{ss} = \varepsilon_{zz} = \gamma_s = \lambda_z = \kappa_{ss} = \kappa_{zz} = 0. \tag{13}$$

Insofar as boundary conditions for the system (10) to (12) are concerned, it is instructive to separately state the conditions for open and closed cross-section tubes. For a tube with open cross-section $s_1 \leq s \leq s_2$ the conditions of no edge tractions are the vanishing of N_{sz}, M_{sz} and P_s (which for a tube with $D_P = A_Q = 0$ reduce and contract to the one condition of vanishing N_{sz}). For a tube with closed cross section we have altogether six conditions of continuity, three of them for N_{sz}, M_{sz} and P_s and three of them, assuming the continuity of displacements, for κ_{zs}, ϵ_{zs} and γ_z (again with reductions and contractions for the case $D_P = A_Q = 0$).

Concerning the form of the sixth-order differential equation system (10) to (12) we have the existence of two first integrals in (10) and (11),

$$N_{sz} = N_0, \qquad \kappa_{zs} = \kappa_0 \tag{14}$$

and the remaining equations in (10) and (11) may be written, with the help of (12), in the form of a fourth order system for N_{zs}, κ_{sz}, Q_z, λ_s, as follows

$$(D\kappa_{sz})' - Q_z + \phi' D_P\lambda_s = -D_*'\kappa_0 \tag{15}$$

$$(D_P\lambda_s)' - N_{zs} - \phi' D\kappa_{sz} = -N_0 + \phi' D_*\kappa_0 \tag{16}$$

$$(AN_{zs})' - \lambda_s - \phi' A_Q Q_z = -A_*' N_0 \tag{17}$$

$$(A_Q Q_z)' - \kappa_{sz} + \phi' AN_{zs} = -\kappa_0 - \phi' A_* N_0. \tag{18}$$

The fourth order system (15) to (18) reduces to a zeroth order system for the case $D_P = A_Q = 0$, as for this case equations (16) and (18) become two equations for N_{zs} and κ_{sz}, without derivatives, and equations (15) and (17) determine Q_z and λ_s without integrations. When $D_P = 0$ and $A_Q \neq 0$ or $A_Q = 0$ and $D_P \neq 0$ a reduction from fourth order to second order occurs.

CALCULATION OF DISPLACEMENTS

Introduction of equations (13) together with the relation $\kappa_{zs} = \kappa_0$ into the strain displacement relations (8) leads to the result that the six components of translational and rotational displacement must be of the form

$$u_z = u_z(s), \qquad \phi_z = \phi_z(s), \qquad \omega = \omega(s) \tag{19}$$

and, except for a state of rigid-body displacements,

$$u_s = \kappa_0 z(y\cos\phi - x\sin\phi), \qquad w = \kappa_0 z(-y\sin\phi - x\cos\phi), \qquad \phi_s = \kappa_0 z. \tag{20}$$

Equations (20) show that the constant of integration κ_0 represents what is usually referred to as the angle of twist of the torsion problem.

TORQUE-TWIST AND UNIVALUEDNESS-OF-DISPLACEMENT RELATIONS FOR CLOSED CROSS-SECTION SHELLS

Writing, on the basis of (8), the relation $\epsilon_{zs}+\epsilon_{sz} = u_{z,s}+u_{s,z}$ and observing that u_z must be a single-valued function of the arclength coordinate s we have, on the basis of the first equation in (20), the displacement univaluedness condition

$$\oint (\varepsilon_{zs}+\varepsilon_{sz})\,\mathrm{d}s-\kappa_0\oint (y\cos\phi-x\sin\phi)\,\mathrm{d}s = 0. \tag{21}$$

Equation (21) is to be used in conjunction with the torque–twist relation (9). Use of the differential equations of equilibrium (10) and appropriate integrations by parts lead to an alternate version of (9), remarkably similar in form to equation (21), namely

$$\oint (M_{zs}+M_{sz})\,\mathrm{d}s+N_0\oint (y\cos\phi-x\sin\phi)\,\mathrm{d}s = T. \tag{22}$$

Equation (22) is effectively equivalent to equation (11) in [2] while equation (21) is effectively equivalent to the differently appearing and differently derived equation (19) in [2].

The principal point of the work in [2] was that use of the two equations equivalent to (21) to (22) would, for the case $A_Q = D_P = 0$, lead to explicit closed-form expressions for N_0 and κ_0, and therewith for stresses and deformations in terms of the applied torque, in such a way that the previously known results for thin-walled closed tubes and for thin-walled open tubes would come out as limiting special cases of formulas providing a continuous transition from one of the special cases to the other.

We obtain a somewhat more general version of our earlier results upon deriving from the constitutive equations (12) with $A_Q = D_P = 0$, and with (14), (10) and (11) (where now $P'_s = \gamma'_z = 0$), expressions for ϵ_{sz}, ϵ_{zs}, M_{sz}, M_{zs} of the form

$$\left(1+\frac{DA}{R^2}\right)\begin{pmatrix}\epsilon_{zs}\\ \epsilon_{sz}\end{pmatrix} = (A+A_*)\left(\frac{1}{1+D(A-A^*)/R^2}\right)N_0+\begin{pmatrix}A\\ A_*\end{pmatrix}\frac{D+D_*}{R}\kappa_0 \tag{23}$$

$$\left(1+\frac{DA}{R^2}\right)\begin{pmatrix}M_{sz}\\ M_{zs}\end{pmatrix} = (D+D_*)\left(\frac{1}{1+A(D-D^*)/R^2}\right)\kappa_0-\begin{pmatrix}D\\ D_*\end{pmatrix}\frac{A+A_*}{R}N_0. \tag{24}$$

Introduction of (23) and (24) into (21) and (22) gives as two simultaneous equations for N_0 and κ_0,

$$N_0\oint (A+A_*)\frac{1+\frac{1}{2}D(A-A_*)/R^2}{1+DA/R^2}\,\mathrm{d}s+\kappa_0\left[S-\oint\frac{(A+A_*)(D+D_*)}{2(1+DA/R^2)R}\,\mathrm{d}s\right] = 0 \tag{25}$$

$$\kappa_0\oint (D+D_*)\frac{1+\frac{1}{2}A(D-D_*)/R^2}{1+DA/R^2}\,\mathrm{d}s-N_0\left[S+\oint\frac{(A+A_*)(D+D_*)}{2(1+DA/R^2)R}\,\mathrm{d}s\right] = \frac{T}{2}. \tag{26}$$

In this S is the cross-sectional area enclosed by the middle surface meridian curve of the shell

$$S = \tfrac{1}{2}\oint (x\sin\phi-y\cos\phi)\,\mathrm{d}s. \tag{27}$$

With AD and $(A+A_*)(D+D_*)$ being of the order of the square of the wall thickness of the shell, equations (25) and (26) are effectively equivalent to the simpler relations

$$N_0 \oint (A+A_*)\,\mathrm{d}s + \kappa_0 S = 0 \tag{28}$$

$$\kappa_0 \oint (D+D_*)\,\mathrm{d}s - N_0 S = \tfrac{1}{2}T. \tag{29}$$

Equations (28) and (29) evidently imply the torque twist relation

$$T = \left\{ 2\oint (D+D_*)\,\mathrm{d}s + \frac{2S^2}{\oint (A+A_*)\,\mathrm{d}s} \right\} \kappa_0 \tag{30}$$

which reduces to our previously given formula for shells uniform in thickness direction, upon setting $A+A_* = 1/2Gh$ and $D+D_* = Gh^3/6$.

The first term on the left in (30) in general is negligible for a closed shell. The case of an open shell is obtained from (30) upon considering an open shell as a special case of a closed shell, with $A+A_* = \infty$ and $D+D_* = 0$ over an appropriate portion of the path of integration. Beyond this, equation (30) evidently allows a continuous transition from the case for which the first term on the right is negligible compared to the second term, to the case for which the second term on the right is negligible compared to the first term. An explicit example of a case where such a continuous transition occurs may be found in [2].

We finally note that explicit generalizations of (23) to (30) are possible by using instead of (12) constitutive equations of the form

$$\begin{aligned} \varepsilon_{zs} &= AN_{zs} + A_* N_{sz} + B\kappa_{zs} + B_* \kappa_{sz} \\ M_{zs} &= D\kappa_{zs} + D_* \kappa_{sz} + CN_{zs} + C_* N_{sz} \end{aligned} \tag{31}$$

with corresponding expressions for ϵ_{sz} and M_{sz}.

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